Functions

If X and Y are sets, a <u>function</u> from X to Y is a relation f such that $\operatorname{dom} f = X$ and $\forall x \in X$ $\exists$ a unique $y \in Y$ s.t. $(x, y) \in f$.

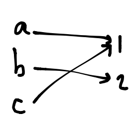
i.e. if $(x,y) \in f$ and $(x,z) \in f$, then $y = z$.

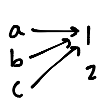
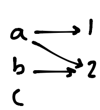
For $x \in X$, the unique y s.t. $(x,y) \in f$ is denoted $f(x)$, called the "value at x"
(standard function notation)

$f: X \to Y$ means "f is a function from X to Y"

domain or source: X; target: Y

Ex:
- The relation $\{(x, x^3 - 1) \mid x \in \mathbb{R}\}$ on $\mathbb{R}$ is the function $f: \mathbb{R} \to \mathbb{R}$ defined $f(x) = x^3 - 1$.
- The function f : set of people $\to$ set of addresses is defined $f(x) =$ address of x
- $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined $f(x) = \dfrac{x^2+1}{x}$ is a function
- $f: \{a, b, c\} \to \{1, 2\}$ defined
  $f(a) = 1$, $f(b) = 2$, $f(c) = 1$ is a function



-  

  function   not a function

- $f: \{x \in \mathbb{Z} \mid 0 \le x \le 100\} \to$ days of 2016
  $f(x) =$ day when the high in Cambridge was $x°F$.

Function? Probably not! (Not "well-defined")

— Not every x corr. to a day

— one x could correspond to multiple days.

- f : days of 2016 $\to \mathbb{R}$

  $f(x)$ = The high temp (in °F) on x is a function

If $f: A \to B$, and $A_0 \subseteq A$, then the <u>image</u> of A_0 under f is

$$f(A_0) = \{ b \in B \mid \exists\, a \in A_0 \text{ s.t. } f(a) = b \}$$

$f(A)$ is called the <u>range</u> or <u>image</u> of f.

If $f(A) = B$, we say f is <u>surjective</u> or <u>onto</u>

<u>Def</u>: • If $f: A \to B$ and A_0 is a subset of A, the <u>restriction</u> of f to A_0 is the function $f|_{A_0} : A_0 \to B$ defined

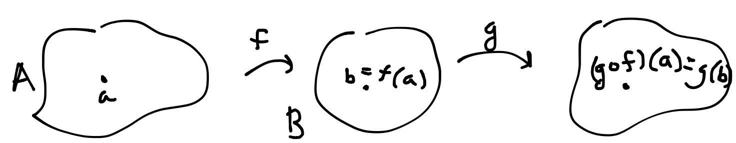
$$f|_{A_0}(a) = f(a).$$

- If $f: A \to B$ and $g: B \to C$, the <u>composite</u> $g \circ f : A \to C$ is defined $(g \circ f)(a) = g(f(a))$.

  Formally, $g \circ f = \{ (a, c) \mid f(a) = b \text{ and } g(b) = c \text{ for some } b \in B \} \subseteq A \times C$

<u>Ex</u>: • $f(x) = x^3$, $g(x) = \dfrac{1}{x^2+1}$, then $(g \circ f)(x) = g(x^3) = \dfrac{1}{x^6+1}$

$(f \circ g)(x) = f\left(\frac{1}{x^2+1}\right) = \frac{1}{(x^2+1)^3}$



<u>Def</u>: A function $f: A \to B$ is <u>injective</u> or <u>one-to-one</u> if $(f(a) = f(b)) \Rightarrow (a = b)$, or, equivalently if $((a \neq b) \Rightarrow f(a) \neq f(b))$

Ex: • $f:\mathbb{R}\to\mathbb{R}$ defined $f(x)=e^x$ is injective (but not surjective).

• $g: \text{set of people} \to \text{days of the year}$ defined $g(x) = \text{birthday of } x$

is not injective (many people share the same birthday), but it is surjective (every date is someone's birthday)

If $A \subseteq \text{set of people}$ is the set of people in this room, then

$g|_A$ is not surjective (there are < 365 of us), but
$g|_A$ may be injective (but unlikely — only about 30% chance)

• If $A_0 \subseteq A$, the function $f: A_0 \to A$ defined $f(a)=a$ is called the inclusion map or embedding of A_0 into A.
An embedding is always injective.

• $f: A\to A$ defined $f(a)=a$ is the identity map and is injective and surjective.

• Define $f: X\times Y \to X$ by $f(x,y)=x$. This is called the projection onto X and is surjective. (When is it injective?)

Def: Let $\sim$ be an equivalence relation on A. Then define the canonical map $f: A \to A/\sim$ to be

$$f(a) = \text{equivalence class of } a.$$

Ex: $x, y \in \mathbb{Z}$ have the same <u>parity</u> if they are both odd or both even (or, equivalently, if $x - y$ is even)

Define the relation $\sim$ to be $x \sim y \iff x$ and y have the same parity. (Check that this is an equiv. relation)

Then $\mathbb{Z}/\sim \; = \left\{ \underbrace{\text{set of odd integers}}_{= \, \overline{1}} , \underbrace{\text{set of even integers}}_{= \, \overline{0}} \right\}$

and the canonical map $c : \mathbb{Z} \to \mathbb{Z}/\sim$ sends $c(-1) = \mathbf{1}$ and $c(100) = \mathbf{0}$

Def: $f : A \to B$ is <u>bijective</u> ("a <u>bijection</u>"), if it is both injective and surjective.

Ex:
- $\begin{tikzcd}[row sep=tiny] a \arrow[r] & 2 \\ b \arrow[r] & 1 \\ c \arrow[r] & 3 \end{tikzcd}$ is a bijection
- $f : \mathbb{R} \to \mathbb{R}$ defined $f(x) = x^3$ is a bijection
- $f : \text{odd integers} \to \text{even integers}$ defined $f(x) = x + 1$ is a bijection.

  (is $f(x) = 2x$??)

Ex: If $f : X \to Y$ is an arbitrary function, we can define an equivalence relation $\sim$ on X:

$$a \sim b \iff f(a) = f(b)$$

(Check that this is an equiv. relation!)

Assume f is surjective. Then we can define a function

$g : Y \to X/\sim$ by $g(y) = \{x \in X \mid f(x) = y\}$

(Why is this well-defined?)

This is in fact a bijection!

Injectivity: Let $y_1, y_2 \in Y$ s.t. $y_1 \neq y_2$.

Since f is surjective, we can find $x_1, x_2 \in X$ s.t.

$f(x_1) = y_1$ and $f(x_2) = y_2$. Thus, $f(x_1) \neq f(x_2)$, so $x_1 \not\sim x_2$.

Since $x_1 \in g(y_1)$ and $x_2 \in g(y_2)$, this means $g(y_1) \neq g(y_2)$, so

g is injective. (On HW: prove by starting w/ $g(y_1) = g(y_2)$)

Surjectivity: Let $\bar{x} \in X/\sim$ ($x \in X$, $\bar{x}$ equiv. class determined by x)

Then $g(f(x)) = \{a \in X \mid f(a) = f(x)\} = \bar{x}$, so g is surjective.

Ex: If $f : X \to Y$ is a set, there is a function

$g : \mathcal{P}(X) \to \mathcal{P}(Y)$ defined $g(A) =$ "$f(A)$"

Notice bad notation!

Lemma: Let $f : A \to B$ and $g : B \to C$ be functions. Then

1.) If f and g are surjective, so is $g \circ f$.

2.) If f and g are injective, so is $g \circ f$.

Pf of 1: (Pf of 2 on hw) $(g \circ f)$ is a function from A to C.

Let $c \in C$. Then since g is surjective, $\exists$ $b \in B$ s.t. $g(b) = c$.

Since f is surjective $\exists$ $a \in A$ s.t. $f(a) = b$. Thus,

$c = g(b) = g(f(a)) = (g \circ f)(a)$, so $(g \circ f)$ is surjective.

Note: This lemma implies that the composition of bijections is also bijective.

Inverse functions

Def: If $f: A \to B$ is bijective $\exists$ a function $f^{-1}: B \to A$ called the inverse of f, defined by letting $f^{-1}(b)$ be the unique $a \in A$ s.t. $f(a) = b$.

Note: Why is f^{-1} well-defined?

If $b \in B$, surjectivity of f implies $\exists$ $a \in A$ s.t. $f(a) = b$.

Injectivity of $f \Rightarrow a$ is unique (i.e. if $f(a) = f(a') = b$, then $a = a'$)

Ex: • The function $f: \mathbb{R} \to \mathbb{R}$ defined $f(x) = x^2$

is not injective ($f(-1) = f(1)$) nor surjective ($f(a) \neq -1$ for any $a \in \mathbb{R}$)

• Let $\overline{\mathbb{R}}_+ = \{x \in \mathbb{R} \mid x \geq 0\} \subsetneq \mathbb{R}$.

Then $g: \overline{\mathbb{R}}_+ \to \mathbb{R}$ defined $g(x) = x^2$ is injective but not surj.

and $h : \mathbb{R} \to \overline{\mathbb{R}}_+$ (defined by restricting target) is surj. but not inj.

$j : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ defined by restricting domain and target is bijective

Its inverse is defined $j^{-1}(x) = \sqrt{x}$.

Ex:

If $f : X \to Y$ is a function, and $A \subseteq Y$, then the inverse image of A is (or preimage)

$$f^{-1}(A) = \{ x \in X \mid f(x) \in A \}$$

Caution! f need not be bijective to define the inverse image.

E.g. If $f : \mathbb{R} \to \mathbb{R}$ is $f(x) = x^2$, then

$$f^{-1}(\overline{\mathbb{R}}_+) = \mathbb{R}, \text{ and } f^{-1}(\mathbb{R}_-) = \emptyset.$$

Claim: Let $f : X \to Y$ and $A \subseteq X$, $B \subseteq Y$.

1.) $f(f^{-1}(B)) \subseteq B$

2.) $A \subseteq f^{-1}(f(A))$

Pf: 1.) If $b \in f(f^{-1}(B))$, then $b = f(a)$ for some $a \in f^{-1}(B)$ and so $f(a) \in B$. $\Rightarrow b \in B$.

2.) If $a \in A$, then $f(a) \in f(A)$,

so $a \in \{ x \in X \mid f(x) \in f(A) \} = f^{-1}(f(A))$.

When does equality hold? See hw!

Claim: Inverse image preserves unions, intersections, and inclusions.

That is, if $f: A \to B$ and $B_0, B_1 \subseteq B$, then

1.) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$

2.) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$

3.) $B_0 \subseteq B_1 \Rightarrow f^{-1}(B_0) \subseteq f^{-1}(B_1)$.